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A Novel Ensemble in Statistical Physics

Igor Volkov,[1,2] Jayanth R. Banavar[1] and Amos Maritan[3]

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We present a theoretical model of a statistical ensemble, in which, unlike in conventional physics, the total number of particles and the energy are not fixed but bounded. It is shown that the temperature and the chemical potential play a dual role: they determine the average energy and the population of the levels in the system and at the same time they act as an imbalance between the energy and population ceilings and the corresponding average values. Different types of statistics (Boltzmann, Bose-Einstein, Fermi-Dirac and one corresponding to the description of a simple ecosystem) are considered. In all cases, we show that the systems may undergo a first or a second order phase transition akin to Bose-Einstein condensation for a non-interacting gas. We discuss numerical schemes for studying the new ensemble. The results of simulations are found to be in excellent agreement with theory.

KEY WORDS: Statistical ensemble, ecology.

1. INTRODUCTION

The conventional grand canonical ensemble in physics describes two systems, one of which ("the reservoir") has many more degrees of freedom than the other ("the system"). They are placed in contact with each other and allowed to exchange both energy and particles. The average values of the energy and number of particles are controlled by the temperature T and the chemical potential μ, respectively. The utility of such an ensemble lies in the fact that it closely represents the conditions under which experiments are often performed.

[1] Department of Physics, The Pennsylvania State University, 104 Davey Laboratory, University Park, PA 16802, USA; e-mail: volkov@psu.edu
[2] Center for Infectious Disease Dynamics, Department of Biology, The Pennsylvania State University, University Park, PA 16802 USA; e-mail: banavar@psu.edu
[3] Dipartimento di Fisica G. Galilei, Università di Padova, via Marzolo 8, I-35131 Padova, Italy; e-mail: amos.maritan@pd.infn.it

Recently, a theoretical scheme for the modification of the grand canonical ensemble has been proposed[1] in the context of an ecosystem. In that ensemble, the system consists of an infinite number of energy levels and is coupled with a reservoir. The total energy and number of particles of both the system and the reservoir are fixed at some predefined values $E_{\max}$ and $N_{\max}$, respectively. Again, the particles and the energy associated with them are permitted to travel back and forth from the system to the reservoir. It was found both theoretically and via simulations that, in equilibrium, the temperature T played a dual role: it controlled the average energy $\langle E \rangle$ of the system (as in the conventional approach) and at the same time it was equal to the imbalance between $E_{\max}$ and $\langle E \rangle$. The chemical potential also had a dual role: apart from controlling the average number of particles $\langle N \rangle$, $T/|\mu|$ served as an imbalance between $N_{\max}$ and $\langle N \rangle$. It should be noted that some other systems, such as the microcanonical quantum ensemble[2], also allow for similar treatment of the controlling parameters.

The above scheme arises naturally in the studies of the dynamics of ecosystems. Indeed, as a first approximation, an ecosystem can be modeled as a community of non-interacting individuals (particles) belonging to different species (energy levels). The concept of finite $E_{\max}$ and $N_{\max}$ arises from the limited resources available to an ecosystem such as space, solar radiation and freshwater supplies.

In the simplest ecologically relevant scenario, the probabilities of the birth and death events (arrival from/departure into the reservoir) in a given level can be chosen to be density independent or proportional to the current population of the species. A non-zero birth rate (speciation) is ascribed to unoccupied levels. This leads to a logarithmic distribution[3] of the number of individuals in each level.

Our earlier theoretical and computational studies of the model of the ecosystem showed that in the absence of constraints on the total population ($N_{\max} = \infty$), an ecosystem may organize in the vicinity of a phase transition akin to Bose-Einstein condensation. The transition is signalled by the macroscopic depletion of the population below a critical temperature.

In this paper, we generalize our previous work by applying our model to systems obeying Bose-Einstein, Fermi-Dirac and Boltzmann statistics and carrying out both theoretical and computational studies. We show that for a system with Bose-Einstein statistics, the results are similar to the previously studied ecology case. Interestingly, the systems obeying Fermi-Dirac or Boltzmann statistics exhibit a first order phase transition and, unlike the ecological or Bose-Einstein cases, this behavior is independent of the density of states. Also, we expand our previous study of ecological systems and show that depending on the value of the birth/death rate ratio there exist three regimes with distinct behaviors.

The outline of the paper is as follows. In Section 2, starting with the one-step master equation, we provide a derivation of the partition function of the new ensemble. For Boltzmann statistics, we demonstrate that the correct counting arises naturally and the Gibbs paradox is averted. In Section 3, we consider different types of statistics and develop the numerical algorithms for simulations. Also, we present some selected results of the simulations and compare them to theoretical expectations. Finally, in Section 4, we discuss the connections between our ensemble and those in classical physics and consider a few examples of our model.

2. THEORETICAL FRAMEWORK

Consider S independent boxes in which balls (particles) can be inserted or removed. We label the boxes (energy levels) using the numbers $1, 2, \ldots, S$. We postulate that the dynamics of the balls is governed by simple, physically motivated rules. Our goal will be to determine the steady state configuration of the system under these rules. Let N represent the total number of balls in the system.

Let us postulate a constant death rate (or removal rate) per ball equal to d_1. Thus the rate of removal of a ball from a given box, with n balls present, is $d_n = d_1 n$ whereas the rate of insertion (which may be thought of as a birth rate) of a ball may be taken generally to be a function, b_n.

Let $P(t; n)$ denote the probability that a given box contains n balls at time t. The time evolution of P is regulated by the master equation[4]:

$$\frac{dP(t;n)}{dt} = P(t;n+1)(n+1)d_1 + P(t;n-1)b_{n-1} - P(t;n)(nd_1 + b_n). \quad (1)$$

The first (second) term on the right hand side corresponds to the removal (insertion) of a ball from the box containing $n+1$ $(n-1)$ particles leading to an enhancement of the probability on the left hand side, whereas the last term corresponds to a depletion of this probability. The stationary solution can be seen to satisfy detailed balance and corresponds to an *equilibrium* situation[4] with

$$P(n) \propto \prod_{m=0}^{n-1} \frac{b_m}{(m+1)d_1}. \quad (2)$$

It follows that, when there are S boxes, all satisfying the same birth-death rules, the unique equilibrium solution is

$$P(n_1, n_2, \ldots n_S) = \prod_{k=1}^{S} P(n_k). \quad (3)$$

One can readily work out other special cases of the framework we have presented. If one chooses $b_n = b_0(n+1)$, one obtains a pure exponential distribution $P(n) = r^n(1-r)$, where $r = b_0/d_1$, which, in turn, leads to the Bose-Einstein distribution[2] for non-degenerate energy levels, i.e. $P(n_1, n_2, \ldots n_S) = r^N(1-r)^S$. On the other hand, if $b_n = 0$ for any n greater than 0 and equal to b_0 otherwise, we find $P(n) = r^n/(1+r)$ for $n = 0$ or 1 and zero for other values of

n and the Fermi-Dirac distribution[2] $P(n_1, n_2, \ldots n_S) = r^N/(1+r)^S$, provided each of the n_i's is 0 or 1 and $P(n_1, n_2, \ldots n_S) = 0$ is zero otherwise.

Note, that the same framework lends itself to the study of the species abundance problem in ecology[1,3,5]. Consider the dynamical rules of birth, death and speciation which govern the population of an individual species. In order to ensure that the community will not become extinct, speciation may be introduced by ascribing a non-zero probability of the appearance of an individual of a new species, i.e $b_0 \neq 0$. If one chooses $b_n = b_0 n$ for $n > 0$ (this amounts to the assumption that the birth rate per individual is constant), one obtains the logarithmic distribution $P(n) = [1 - \ln(1-r)]^{-1} r^n/n$ which, in turn, leads to the well-known Fisher log-series distribution[6], i.e. $\langle \phi_n \rangle = \theta r^n/n$, where $\theta = S/[1 - \ln(1-r)]$ and $n > 0$. Here, $\langle \phi_n \rangle$ represents the average number of species (boxes) with population n.

If $b_n = b_0$ is taken to be constant, one finds the Poisson distribution $P(n) = e^{-r} r^n/n!$. This leads to $P(n_1, n_2, \ldots n_S) \propto r^N / \prod_{k=1}^{S} n_k!$, which is the celebrated Boltzmann counting in physics in the grand canonical ensemble and where r plays the role of a fugacity and $N = \sum_k n_k$. It is noteworthy that, unlike in conventional classical treatments[2] in which one obtains an additional factor of $N!$, here one gets the correct Boltzmann counting and one avoids the well-known Gibbs paradox in this scheme. If one were to ascribe energy values ε_k to each of the boxes and enforce a fixed average total energy, one would get the standard Boltzmann result that the probability of occupancy of an energy level ε is proportional to $e^{-\beta\varepsilon}$, where β is proportional to the inverse of the temperature.

Now let us assign the energy ε_k to the k-th level (box) so that $0 < \varepsilon_0 < \varepsilon_1 < \ldots$, and introduce the constraints $E_{\max}$ and $N_{\max}$ on the total energy and population of the system.

The partition function for the system with fixed $E_{\max}$ and $N_{\max}$ may be written as

$$Q = \sum_{\{n_k\}} \prod_k P(n_k)\, \Theta\,(E_{\max} - \varepsilon_1 n_1 - \varepsilon_2 n_2 - \ldots)\, \Theta\,(N_{\max} - n_1 - n_2 - \ldots), \quad (4)$$

where $\Theta(x)$ is the unit step function, equal to 0 for $x < 0$ and 1 for $x \geqslant 0$.

Using the integral representation for Θ function[7] one can rewrite the above equation in the following form:

$$\begin{aligned} Q &= \sum_{\{n_k\}} \prod_k P(n_k) \int_{\gamma_1} \frac{dz_1}{2\pi i z_1} e^{z_1(E_{\max} - \varepsilon_1 n_1 - \varepsilon_2 n_2 - \ldots)} \int_{\gamma_2} \frac{dz_2}{2\pi i z_2} e^{z_2(N_{\max} - n_1 - n_2 - \ldots)} \\ &= \int_{\gamma_1} \frac{dz_1}{2\pi i z_1} e^{z_1 E_{\max}} \int_{\gamma_2} \frac{dz_2}{2\pi i z_2} e^{z_2 N_{\max}} \prod_k \sum_{n=0}^{\infty} P(n) e^{-(z_2 + z_1 \varepsilon_k) n}, \end{aligned} \quad (5)$$

where the contours $\gamma_{1,2}$ are parallel to the imaginary axis with all their points having a fixed real part $x_{1,2}$ (i.e. $z_{1,2} \in \gamma_{1,2} \Leftrightarrow z_{1,2} = x_{1,2} + iy_{1,2}, -\infty < y_{1,2} < +\infty$). The integral is independent of $x_{1,2}$ provided $x_{1,2}$ is positive[7].

Let

$$e^{-f(x)} = \sum_{n=0}^{\infty} P(n)e^{-xn}. \tag{6}$$

Then

$$Q = \int_{\gamma_1}\int_{\gamma_2} dz_1 dz_2 e^{g(z_1,z_2)}, \tag{7}$$

where

$$g(z_1, z_2) = -\ln(z_1) - \ln(z_2) + z_1 E_{\max} + z_2 N_{\max} - \sum_k f(z_2 + z_1\varepsilon_k). \tag{8}$$

In order to evaluate the integral in Eq. (7) we will apply the steepest descent method[7]. Let us expand $g(z_1, z_2)$ about the point $(x_1 + i0, x_2 + i0)$, where it is maximum:

$$g(z_1, z_2) \approx g(x_1, x_2) - \frac{1}{2!} g_{z_1^2}(x_1, x_2) y_1^2 - \frac{1}{2!} g_{z_2^2}(x_1, x_2) y_2^2 - g_{z_1,z_2}(x_1, x_2) y_1 y_2, \tag{9}$$

where $g_{z_1^{k_1}, z_2^{k_2}} \equiv \frac{\partial^{k_1+k_2} g(z_1,z_2)}{\partial z_1^{k_1} \partial z_2^{k_2}}$. Because $g(x_1, x_2)$ is a maximum,

$$g_{z_1}(x_1, x_2) = g_{z_2}(x_1, x_2) = 0. \tag{10}$$

Substituting this expression into Eq. (7) and performing the integration one obtains

$$Q = \frac{e^{g(x_1,x_2)}}{\sqrt{g_{z_1^2}(x_1, x_2) g_{z_2^2}(x_1, x_2) - g^2_{z_1,z_2}(x_1, x_2)}} = \frac{\exp\left[\frac{E_{\max} - \mu N_{\max}}{T} - \sum_k f\left(\frac{\varepsilon_k - \mu}{T}\right)\right]}{\sqrt{1 - \sum_k \frac{\varepsilon_k^2 + \mu^2}{T^2} f''\left(\frac{\varepsilon_k - \mu}{T}\right)}} \tag{11}$$

Here we neglect the term $\mu^2/(2T^4) \sum_{k \neq k'} f''\left(\frac{\varepsilon_k - \mu}{T}\right) f''\left(\frac{\varepsilon_{k'} - \mu}{T}\right)(\varepsilon_k - \varepsilon_{k'})^2$ under the square root in the denominator, omit the constant factor in the expression for Q and replace x_1 and x_2 by $1/T$ and $-\mu/T$, respectively ($T \geqslant 0$ and $\mu \leqslant 0$). Eq. (10) yields

$$E_{\max} = T + \sum_k \varepsilon_k f'\left(\frac{\varepsilon_k - \mu}{T}\right) \tag{12}$$

and

$$N_{\max} = -\frac{T}{\mu} + \sum_k f'\left(\frac{\varepsilon_k - \mu}{T}\right). \tag{13}$$

The free energy, F, is given by

$$F \equiv -T \ln Q = -E_{\max} + \mu N_{\max} + F_0 + F_1, \tag{14}$$

where

$$F_0 = T \sum_k f\left(\frac{\varepsilon_k - \mu}{T}\right) \tag{15}$$

and

$$F_1 = \frac{1}{2} T \ln\left[1 - \sum_k \frac{\varepsilon_k^2 + \mu^2}{T^2} f''\left(\frac{\varepsilon_k - \mu}{T}\right)\right]. \tag{16}$$

The last term, F_1, can be neglected for large enough systems because $F_1/F_0 \propto \ln(V)/V \ll 1$, where V is the characteristic size of the system.

Using Eq. (14), one can find the relative entropy S:

$$S = -\frac{\partial F}{\partial T} = -\sum_{k,n} \widetilde{P}_{n,k}(T) \ln \frac{\widetilde{P}_{n,k}(T)}{P_n}, \tag{17}$$

where

$$\widetilde{P}_{n,k}(T) = \frac{P_n e^{-(\varepsilon_k - \mu)n/T}}{\sum_m P_m e^{-(\varepsilon_k - \mu)m/T}}. \tag{18}$$

The average population of the k-th level is given by

$$\langle n_k \rangle = \frac{\partial F}{\partial \varepsilon_k} = f'\left(\frac{\varepsilon_k - \mu}{T}\right) \tag{19}$$

and the average population and the total energy of the system are defined as $\langle N \rangle = \sum_k \langle n_k \rangle$ and $\langle E \rangle = \sum_k \varepsilon_k \langle n_k \rangle$, respectively. Thus one can rewrite Eqs. (12) and (13) as

$$E_{\max} = T + \langle E \rangle \tag{20}$$

and

$$N_{\max} = -\frac{1}{\ln(\alpha)} + \langle N \rangle, \tag{21}$$

where $\alpha \equiv \exp(\mu/T)$.

The above equations demonstrate the dual role of the temperature and the chemical potential, which was discussed earlier and represent the central result of our derivation.

Let us postulate that $\varepsilon_k = k^{1/(d+1)}$ with $d > -1$, i.e. the number of energy levels per unit energy interval (the density of states) scales as $V\varepsilon^d$, where V is the size of the system. As in conventional statistical physics, a true phase transition can only occur in the thermodynamical limit, when both N and V become very large and the density $N_{\max}/V$ is fixed. In what follows we will work in units in which V is set equal to 1.

In a continuum formulation, one obtains the following expressions for the average energy $\langle E\rangle$ and population $\langle N\rangle$ of the ecosystem:

$$\langle E\rangle = T^{d+2} I_1(\alpha) \tag{22}$$

and

$$\langle N\rangle = T^{d+1} I_0(\alpha), \tag{23}$$

where

$$I_m(\alpha) = (d+1)\int_0^\infty f'[t - \ln(\alpha)]t^{d+m}\,dt. \tag{24}$$

In the next section we will show, by explicitly calculating the integrals in Eq. (24), that for systems with Bose-Einstein, Fermi-Dirac and Boltzmann statistics, $I_{0,1}(\alpha)$ are 0 when $\alpha = 0$ and monotonically increase as α approaches 1.

From Eqs. (20)–(23) one can see that, for a given $E_{\max}$, the temperature T cannot become lower than some value $T_{\min}$, which occurs when there is no constraint on the total population, i.e $N_{\max} = \infty$. Similarly, the system reaches the maximum $T = E_{\max}$ when $N_{\max} = 0$ and the system is empty.

In order to analyze whether the system can undergo a phase transition, we will use a scheme similar to the familiar one in Bose-Einstein condensation[8]. Let us fix the value of $N_{\max}$ in Eq. (21) and vary the temperature (this can be done by varying $E_{\max}$). At very high temperatures, the values of α are very small and thus one can neglect the first term (imbalance) in the rhs of Eq. (21). This means that the system is populated to its full capacity $N_{\max}$. As we decrease the temperature, α approaches 1 and the imbalance can no longer be neglected (the system undergoes depletion). If $I_0(1)$ is finite then one can introduce a critical temperature T_c

$$T_c = \left(\frac{N_{\max}}{I_0(1)}\right)^{\frac{1}{d+1}} \tag{25}$$

above which $\langle N\rangle \approx N_{\max}$ and below which the system undergoes a rapid depletion. Note that in our ensemble the imbalance $-1/\ln(\alpha)$ acts as a zero groundstate level in the conventional grand canonical ensemble: the macroscopic depletion of the population of the former is analogous to the macroscopic occupation of the groundstate of the latter.

Finally, let us note that $\alpha = 1$ is related to two cases: first, it enters the expression for T_c and, second, $\alpha = 1$ when the system has just the $E_{\max}$ constraint

($N_{\max} = \infty$). This suggests that $T_{\min} = T_c$ provided that T_c exists (i.e. $I_0(1)$ is finite). Indeed, let us consider the following scenario: when $N_{\max} = \infty$ the system organizes at $T_{\min}$ with some average population N_M and from Eq. (23) it follows that $T_{\min} = \left(\frac{N_M}{I_0(1)}\right)^{\frac{1}{d+1}}$. But this is also the critical temperature for the system with $N_{\max} = N_M$. Note that in the simulations the actual transition temperature T_c' slightly differs from T_c since for finite $N_{\max}$ the value of α cannot reach 1.

3. THEORETICAL AND NUMERICAL RESULTS FOR SYSTEMS WITH DIFFERENT STATISTICS

We now consider four distinct cases and demonstrate that the behavior observed in simulations is in excellent accord with the theoretical predictions.

3.1. Boltzmann Statistics (Figures 1 and 2)

3.1.1. Theory

For Boltzmann statistics ($b_n = b_0,\ d_n = d_1 n$ and $f(x) = r[1 - \exp(-x)]$) Eqs. (20) and (21) give

$$\langle n_k \rangle = \alpha r e^{-\varepsilon_k / T}, \tag{26}$$

$$E_{\max} = T + (d+1)\Gamma(d+2)\alpha r T^{d+2} \tag{27}$$

and

$$N_{\max} = -\frac{1}{\ln \alpha} + \Gamma(d+2)\alpha r T^{d+1}, \tag{28}$$

where $r = b_0/d_1$ and $\Gamma(x) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function.

One can see that $I_0(1)$ is finite for any d. Thus the system with Boltzmann statistics can undergo a first order phase transition (see Figure 2).

From Eq. (26) it follows that the average population of any level cannot exceed r, hence r should be large (we used $r = 100$ in simulations).

3.1.2. Simulations

At any given time step, a level is randomly picked and a random number R in the interval [0, 1) is generated. If $R < b_0/(n+1)$ and there is sufficient energy and available particles in the reservoir a birth event occurs (here n is the occupancy of the level). If $R \geqslant 1 - d_1$ and the level is occupied, a death event occurs. Otherwise no action is taken. Note that in this scheme the birth and death rates are chosen to be $b_n = b_0/(n+1)$ and $d_n = d_1$ (this is equivalent to $b_n = b_0$ and $d_n = d_1 n$ since only the ratio b_n/d_{n+1} enters the expression for the probability). This choice is

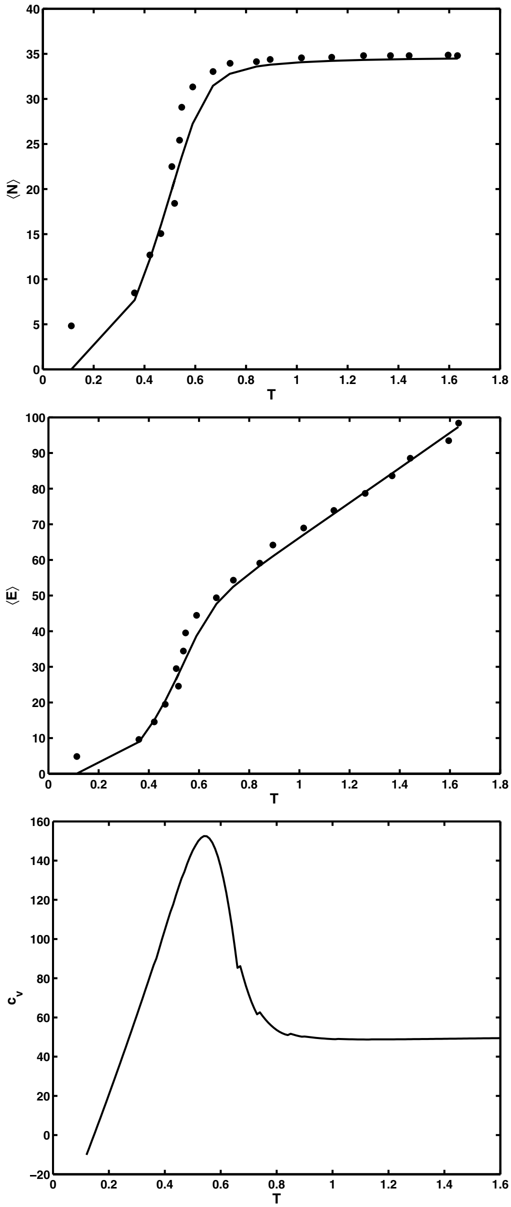

Fig. 1. The results of the simulations of the novel ensemble with Boltzmann statistics. $r = 100$. $\varepsilon_k = k^{2/3}$, $k = 1..1000$, $N_{\max} = 35$, $T_{\min} \approx 0.62$ (one can use Eq. (20) to determine the value of $E_{\max}$ associated with the transition temperature $T_{\min}$). Here $C_v = \partial\langle E\rangle/\partial T$ is the specific heat of a system. The peak in the specific heat occurs at the phase transition. The solid line denotes the theoretical prediction.

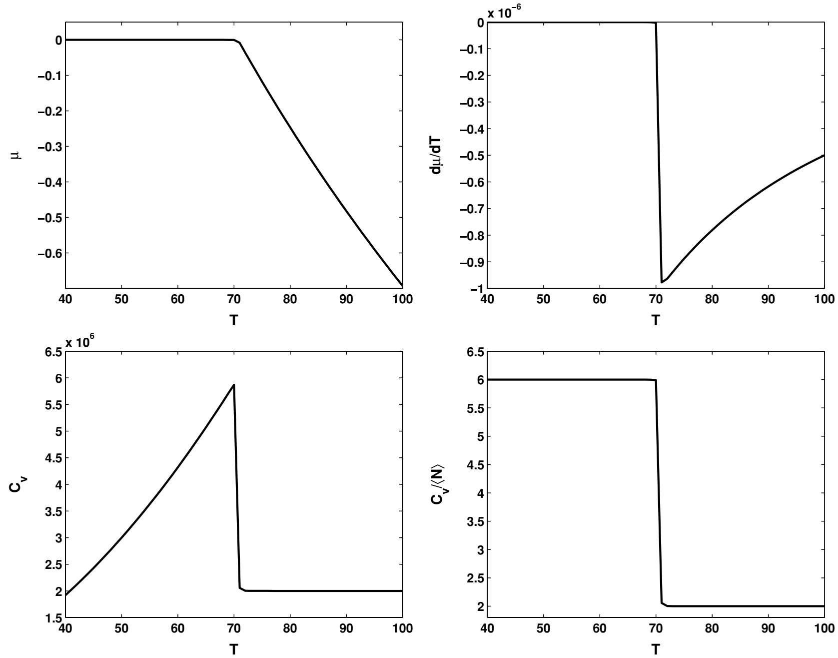

Fig. 2. Boltzmann Statistics. $r = 100$, $d = 1$, $N_{\max} = 10^6$.

helpful because both b_n and d_n are finite for arbitrary n. However the dynamics of cases with the same ratio b_n/d_{n+1} but different choices for the death and birth rate are different (for example relaxation times etc.). This is true for the other cases considered below. The absolute value of b_0 (recall that $d_1 = b_0/r$) is not important provided that $b_n + d_n \leqslant 1, \forall n$.

3.2. Fermi-Dirac, Bose-Einstein and Ecological Systems

The above analysis can be straightforwardly applied to Fermi-Dirac and Bose-Einstein statistics as well as the ecologically relevant case. We have found that systems with Fermi-Dirac statistics behave exactly in the same way as the one with Boltzmann statistics. However, Bose-Einstein systems and ecological systems may exhibit quite different behaviors depending on the values of parameters r and d:

When $r < 1$, the system is underpopulated, i.e. $\langle n_k \rangle$ is finite for any value of T and the system undergoes a phase transition for any value of $d > -1$,

When $r = 1$ the system can undergo a continuous phase transition only for a class of density of states with $d > 0$ (see Figures 3 and 4).

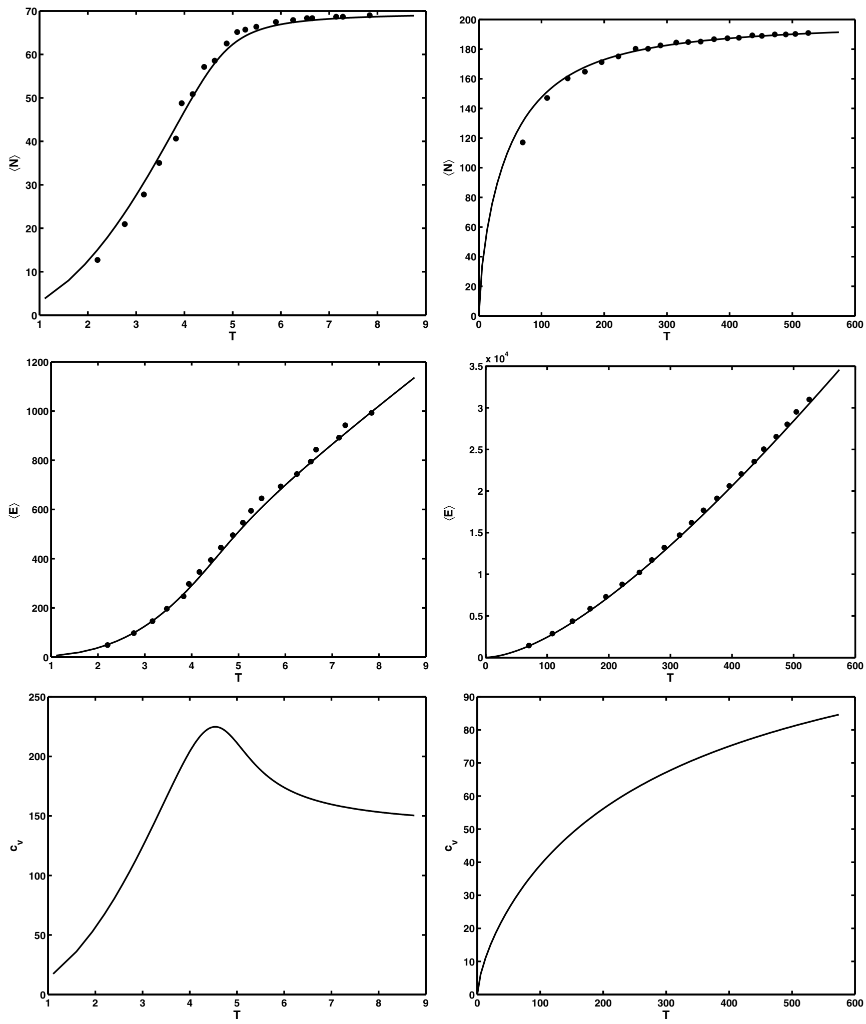

Fig. 3. The results of the simulations of the novel ensemble with Bose-Einstein statistics. $r = 1$. Left panel: $\varepsilon_k = k^{0.5}$, $k = 1..100, 000$, $N_{\max} = 70$, $T_{\min} \approx 4.87$. Right panel: $\varepsilon_k = k^{3/2}$, $k = 1..1000$, $N_{\max} = 275$, $T_{\min} \approx 115$. The peak in the specific heat occurs at the phase transition. Note the absence of a phase transition when d is negative. The solid line denotes the theoretical prediction.

Finally, when $r > 1$, one finds, generally, for sufficiently large $E_{\max}$ (and $N_{\max} = \infty$) that the occupancy of the excited levels is small and independent of $E_{\max}$ with the population of the ground state increasing proportional to $E_{\max}$. The behavior of the system is qualitatively independent of d. Note that the saddle point approximation does not hold for this case.

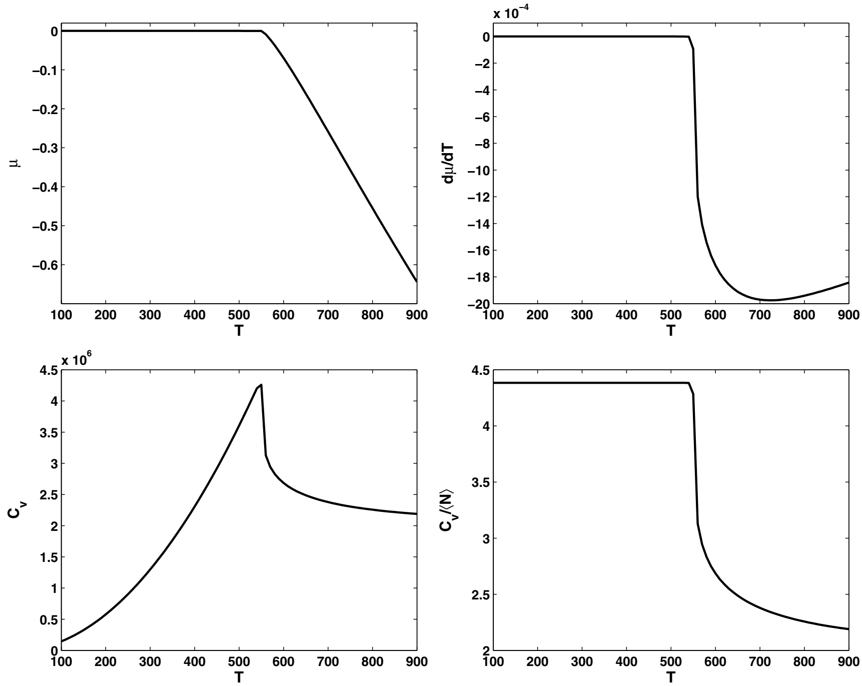

Fig. 4. Bose Statistics. $r = 1$, $d = 1$, $N_{\max} = 10^6$.

Also, in ecological systems, one would expect, in the simplest scenario, that there ought to be a co-existence of all species in our model with an infinite population of each when there are no constraints whatsoever or equivalently when $E_{\max} = N_{\max} = \infty$. This case corresponds to $r = 1$.

4. CONCLUSIONS

We conclude with a brief discussion on the relationship of the novel ensemble that we have studied here with standard ensembles in physics. The familiar microcanonical ensemble is obtained on replacing both Θ functions in Eq. (4) by Dirac delta functions. Indeed, by implementing the steepest descent method on $E_{\max}$ term, one obtains the canonical ensemble whereas on using the steepest descent method on both the $E_{\max}$ and $N_{\max}$ terms, one obtains the grand canonical ensemble.

Our numerical scheme is readily modified for the study of the canonical ensemble. At each timestep, two events occur (provided that the energy constraint is satisfied): two levels are chosen randomly and a death of a particle in one level is followed by a birth of the particle in another, provided that the energy constraint is satisfied (otherwise no action is taken). No action is taken if the first level chosen has no particles in it. We have implemented this scheme and confirmed that it is in excellent accord with theoretical expectations.

As a possible application of our model, one can consider the effect of photoexcitation (and/or photoionization), which occurs when the radiation produced when an external source interacts with the surrounding atomic gas (e.g., planetary nebulae or OB star associations embedded in gas clouds[9]. In this case, the processes of birth/death are represented by excitation/deexcitation (ionization/recombination). The maximum number of electrons that can possibly go into excited states (or, in the case of ionization, leave the atom) corresponds to $N_{\max}$ and the radiation flux can be associated with $E_{\max}$. One would expect then that the stimulated emission from the gas will follow the phase transition scenario described in this paper, i.e. on decreasing $E_{\max}$ below some critical value one would observe a rapid decline in the flux of stimulated emission.

A more direct example of our ensemble is a shopping game. Consider a consumer shopping in a supermarket. The energy levels correspond to the different types of products (the products are distinguished from each other by their price only). The total amount of money that the consumer has corresponds to $E_{\max}$. The analog of $N_{\max}$ is the limit on the maximum number of items that the consumer could buy and is determined, say, by the size of the consumer's shopping cart. The dynamics of the game consists of the following rules. The analog of birth is selection of an item from the store shelf and adding to the cart provided that the number of items in the cart does not exceed the threshold and provided that the shopper has sufficient money to buy all the merchandize in the cart. The removal of an item from the cart and returning it to the shelves corresponds to a death event.

Let us reformulate the rules discussed earlier in the language of this shopping game. For Boltzmann statistics, the addition event corresponds to the placement of an item of a randomly picked product in the cart and the death event is the removal of a random item from the cart. For Bose-Einstein statistics, the addition event is the same as for Boltzmann statistics and the death event corresponds to the removal of an item of a randomly picked product already in the cart. For this case, r is a measure of the ratio of addition to removal attempts. Fermi statistics has the same rules as Bose-Einstein statistics with the constraint that at most there is just one item in the cart of any given product. The ecology case consists of addition of an item of a product already contained in the cart with a probability proportional to the number of such items, a non-zero probability of the addition of an item of a product not already represented in the cart and the removal of a randomly picked item present in the cart. Note that the above rules are not unique and there are many ways to obtain any desired statistics.

Our two key results can be stated as follows. First, the average quantity of money remaining in the shopper's wallet (the imbalance) is non-zero and determines the relative numbers of items of different products represented in the cart. The imbalance magnitude plays the role of temperature in the system. For a given $E_{\max} = E_M$, with no $N_{\max}$ constraint, let the average number of items in the cart be denoted by N_M. The novel transition that we observe corresponds to the case in which $N_{\max} = N_M$ and occurs on varying $E_{\max}$ or the total money in the wallet. There is a sharp depletion in the number of items in the cart as $E_{\max}$ drops below E_M. Interestingly this phase transition occurs for any density of states for the Boltzmann and Fermi-Dirac cases and is a first order but only for the 'right' density of states, when $r = 1$, for the ecology and Bose-Einstein cases, where it becomes a continuous transition.

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